Energy & Universal Gravitation

In a constant gravitation field, the potential energy due to gravity is simply $U_g = mgh$. Up until now, we have done problems that were all basically close to the ground, where that is a very useful, and reasonably accurate approximation. Now that you know that the gravitational field around an object is actually not constant, I hope you realize this means the "mgh" is no longer valid and wildly incorrect. So let's figure out the correct expression for potential energy!

Remember that potential energy is just the energy "stored" by a conservative force when something moves from one place to another fighting that force. Officially, one would say that the potential energy gained or lost would be equal to the negative of the work done by the conservative force moving from one place to another:

$$\Delta U = -W = -\int F\,dx$$

Before we do the new equation for gravitational potential energy, let's re-derive our two old equations for potential energy, $U_g = mgh$ and $U_s = 1/2\,kx^2$, but in a slightly different way than before. This may seem repetitive and annoying, but it will help make our final derivation make more sense.

First, calling "up" positive y, the potential because of a uniform gravitational force $F = -mg$ becomes

$$U = -W = -\int(-mg)\,dy = mgy + C$$

We have a constant of integration to deal with. It would be lovely to call it 0 when the potential is 0, so let's see what value of y lets that happen:

$$U = mgy + C \longrightarrow 0 = mgy + 0 \longrightarrow y = 0$$

So our old equation is $U_g = mgy$, though we usually used h instead of y, and zero potential energy is defined when the height is zero, just like we have done for months. Now let's do the same thing for a spring, where $F = -kx$:

$$U = -W = -\int(-kx)\,dx = \tfrac{1}{2}kx^2 + C$$

And like before, let's find the x value that let's the constant of integration be equal to zero when there is zero potential energy:

$$U = \tfrac{1}{2}kx^2 + C \longrightarrow 0 = \tfrac{1}{2}kx^2 + 0 \longrightarrow x = 0$$

Again, we have our old equation, with the upstretched spring corresponding to zero potential energy.

Finally, let's do something new. Gravity is not really a uniform force field, so using Newton's Universal Gravitation formula $F = -Gm_1m_2/r^2$ we have the potential energy given by:

$$U = -W = -\int\left(-\frac{Gm_1m_2}{r^2}\right)dr = -\frac{Gm_1m_2}{r} + C$$

As before, let's find the r that lets the constant C be zero when the potential energy is zero:

$$U = -\frac{Gm_1m_2}{r} + C \longrightarrow 0 = -\frac{Gm_1m_2}{r} + 0 \longrightarrow r = \infty$$

So here is our shiny new equation for gravitational potential energy between two masses separated by a distance r:

$$\boxed{U = -\frac{Gm_1m_2}{r} \text{ with zero potential energy defined when the two masses are infinitely far apart}}$$

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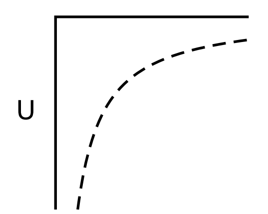

Yes, that means potential energy is negative when dealing with universal gravitation. As weird as it first seems, its actually logical and you will get used to it. Note that the potential energy between two objects does increase as they get further apart, and decreases as they get closer, just like you would expect. This is perhaps easier to see in the graph to the right. Apart from the negative, the biggest impact on problem solving is that when using energy conservation to analyze a situation, there will almost always be both an initial and a final potential energy. (The only exception is when the objects either start or end infinitely far away, which makes that potential energy equal to zero.)

Let's do an example:

Imagine a rock, mass m, is launched straight up with a speed of 1000 m/s from the surface of the moon (mass M and radius R.) How far away from the center of the moon, r, does it go?

By conservation of energy we say that the initial energy of the rock equals the final energy of the rock.

$$U_i + K_i = U_f$$

$$-\frac{GmM}{R} + \tfrac{1}{2}mv^2 = -\frac{GmM}{r}$$

$$\frac{1}{r} = \frac{1}{R} - \frac{v^2}{2GM}$$

Let's plug in the numbers now, as that will rapidly become a blob:

$$\frac{1}{r} = \frac{1}{1.74x10^6} - \frac{1000^2}{2\left(6.67x10^{-11}\right)\left(7.3x10^{22}\right)} \longrightarrow r = 2.12x10^6$$

The rock would end up 2120 km away from the center of the moon - which would be 380 km above the surface of the moon.

Escape Speed

Now let's extend these energy ideas with an abstract question: how much work would it take to "lift" an object infinitely far away? We will do this two ways, and will use m for the mass of the object, M for the mass of the planet and R for the radius of the planet, where the object starts. Lifting also implies we start and end at rest.

Basic energy ideas means the system has some initial energy and work is done on it to give it a final energy. Therefore we can say (and remember that the final potential energy when infinitely far away is 0):

$$U_i + W = U_f \longrightarrow -\frac{GmM}{R} + W = 0 \longrightarrow W = \frac{GmM}{R}$$

We could also directly use the definition of work and do out the integral with the assumption that our force always exactly cancels out gravity, so that we don't do any extra work to lift it:

$$W = \int_R^\infty \frac{GmM}{r^2}dr = -\frac{GmM}{r}\bigg|_R^\infty = -\frac{GmM}{\infty} - \left(-\frac{GmM}{R}\right) = \frac{GmM}{R}$$

Perhaps it seems a little strange at first, but it would take a finite amount of energy to lift something infinitely far away. That means if something were "dropped" from infinitely far away, it would gain a finite amount of kinetic energy. How fast?

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$$U_i = U_f + K_f \longrightarrow 0 = -\frac{GmM}{R} + \tfrac{1}{2}mv^2 \longrightarrow v = \sqrt{\frac{2GM}{R}}$$

For something falling into the earth from infinity, that turns into

$$v = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{2\left(6.67x10^{-11}\right)\left(6x10^{24}\right)}{\left(6.4x10^{6}\right)}} = 11{,}200 m/s$$

Instead of asking how much work to lift an object infinitely far away, we could have asked how much kinetic energy (and thus what speed) does an object need so that it will "rise" infinitely far away? It's the same calculation and answer:

$$U_i + K_i = U_f \longrightarrow -\frac{GmM}{R} + \tfrac{1}{2}mv^2 = 0 \longrightarrow v = \sqrt{\frac{2GM}{R}}$$

Worded that way it is more obvious what is special about that speed. It is the minimum speed an object would need to be "thrown" so that it goes infinitely far away, in other words, that is the *escape speed*. Since your book lists it as a special equation, we will put it in a box:

$$\boxed{v_e = \sqrt{\frac{2GM}{r}}}$$

Notice how there is a small r in the equation and not a capital R. That is because one can talk about the escape speed of an object when it is a distance r away from a mass M; it doesn't have to from the surface of a planet. (Also notice how the mass of the object escaping doesn't matter - it canceled out in the energy equations.)

For example, the moon moves at about 1000 m/s as it travels around the earth. How fast would it have to move so that it escaped the earth? Using the mass of the earth (what the moon is escaping from) and the radius of the moon's orbit (how far away the moon is from the earth) we get

$$v_e = \sqrt{\frac{2GM}{r} = \sqrt{\frac{2\left(6.67x10^{-11}\right)\left(6x10^{24}\right)}{3.84x10^{8}}}} = 1444 m/s$$

This is significantly less than the 11,200 m/s escape speed from the surface of the earth.

Orbits

Let's examine some energy ideas when an object orbits a much larger object, like satellites around a planet or planets around a star. A circular orbit is pretty easy to understand: an object (m) orbits around a large body (M) in a circle with a constant radius (R) and with a constant speed (v). Hopefully, it is obvious that the energy (E) of the orbit is constant, and would be given by

$$E = K + U = \tfrac{1}{2}mv^2 - \frac{GmM}{R}$$

We also know that the net force on the smaller mass is a centripetal force and is a result of the gravitational attraction between the two bodies, so that we can also say

$$\sum F = ma \longrightarrow \frac{GmM}{R^2} = \frac{mv^2}{R} \longrightarrow v^2 = \frac{GM}{R}$$

Plugging that expression for v^2 into the energy equation gives us

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$$E = \tfrac{1}{2}m\left(\frac{GM}{R}\right) - \frac{GmM}{R} = -\frac{GmM}{2R}$$

Like most of these derivations, even though we used the special case of a circular orbit, the equation holds true even for an elliptical orbit, where R is the semimajor axis of the orbit. Again, let's put it in a box as it is one of our big equations for the unit:

$$\boxed{E = -\frac{GmM}{2R}}$$

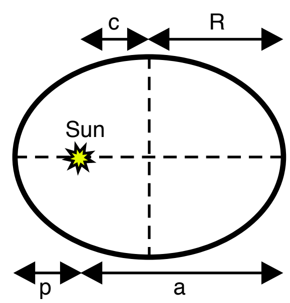
The energy of an orbit is constant, and is negative. To help see why that makes sense, imagine the speed of the smaller object is the escape speed of the object. Then the energy of the system is

$$E = K + U = \tfrac{1}{2}mv^2 - \frac{GmM}{R} = \tfrac{1}{2}m\left(\sqrt{\frac{2GM}{R}}\right)^2 - \frac{GmM}{R} = 0$$

If the total energy of the two objects is zero (or greater) there won't be an orbit and the two masses are not bound together. In this case, the "orbit" of the smaller object would be a parabola if E = 0, or a hyperbola if E > 0. Many comets that come by the sun have an energy greater than zero - which means they swing by the sun once, and then never come back.

Satellites in orbit around the earth have a negative total energy because they are bound in the orbit. Satellites that are launched to examine places other than the earth are given total energies that are greater than zero so they can escape the earth. Many satellites have been launched with positive energies with respect to the sun, which means that they will leave the solar system and never return.

The diagram to the right shows an elliptical orbit around the sun. The dashed lines are the two axes of the ellipse, and the sun is at one of the foci. R is therefore the semi-major axis and c is the distance from the center of the orbit to the sun. By definition, the *eccentricity* of the ellipse is $e = c/R$. Along the major axis, the closest an object would ever get to the sun is *perihelion*, and is marked with a p. The farthest is marked a and is called *aphelion*. A planet would be going the fastest at perihelion and the slowest at aphelion. At all times, its energy would be constant; when it loses potential energy by getting closer to the sun it gains kinetic energy and goes faster. The opposite is true when getting farther from the Sun. In general, for an elliptical orbit we can say



$$E = K + U \longrightarrow -\frac{GmM}{2R} = \tfrac{1}{2}mv^2 - \frac{GmM}{r}$$

In that expression, v is the speed of the planet when it is a distance r from the sun. Notice that this equation nicely shows the planet will move the fastest when it is the closest to the sun and its slowest speed occurs at its maximum distance from the sun.

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